Computational Dynamics of Solid Waste Generation and Treatment in the Presence of Population Growth

J. P. SENZIGE, O.D. MAKINDE, K. NJAU, Y. NKANSAH-GYEKYE

Abstract. Solid waste management is a global challenge and the situation is worse in urban areas of the developing countries where, in most cases, there are no data of how much solid waste is generated over a specific period of time. Worse still, solid waste continues to pile up as population continues to grow and authorities in developing countries are increasingly becoming unable to manage the waste. This calls for a means for anticipating the solid waste to be generated in order for the authorities to take proactive actions in managing the solid waste. In this paper, a deterministic compartmental mathematical model that can be used to predict the amount of solid waste generation and treatment needed with population growth as major factor is proposed and analysed qualitatively using the stability theory of differential equations. Numerical simulation is performed to validate the qualitative results.

1. Introduction

The history of solid waste generation dates back to ancient times. Humans began producing solid waste when they first settled down into small non-nomadic communities at around 10,000 BC (Worrel and Vesilind, 2012). These small communities managed to bury solid waste generated just outside their settlements or disposed them in nearby rivers and water bodies (Marshall and Farahbakhsh, 2013, Seadon, 2006) as, at such times, most of the solid waste produced was essentially organic. But as the small communities evolved into large and sophisticated communities, the solid waste produced also increased, both in quantity and type. Thus burying and disposing then in rivers were no longer feasible options and hence the need for solid waste management. Maribel Cruz defines solid waste management as an interrelated system of appropriate technologies and mechanisms involved in the generation, collection, storage and processing, transfer and transport, and disposal of solid waste at the lowest possible cost and risk to health of the people and their environment. The first organised solid waste management practices were implemented in the ancient city of Mahenjo-Daro in the Indus Valley by 2000 BC (Worrel and Vesilind, 2012). By 500 BC, the Greeks had both issued a decree banning waste disposal in the streets and setup the first Western-type ‘municipal dumps’ while the Chinese had ‘disposal police’ to enforce disposal laws by 200 BC (Melosi, 1981). As noted by Henry et al (2006), Wilson (2007) and Nemerow (2009) the primary purpose for solid waste management strategies is to address the health, environmental aesthetic, land use, resource, and economic concerns associated with improper solid waste management.
Although solid waste management is an important environmental challenge the world over, the problem is bigger in the developing world (Guerrero et al, 2013, Marshal and Farahbakhsh, 2013, Joseph, 2002; ADB, 2002; Aancheng, 2003) where the waste produced by burgeoning cities is overwhelming local authorities and national governments alike (Tacoli, 2012; Yousif and Scott, 2007). As a result, the uncollected waste, which is often mixed with human and animal excreta, is dumped indiscriminately along the streets, in river valleys and in drains, so contributing to flooding, breeding of insect and rodent vectors and the spread of diseases (UNEP-IETC, 1996). We all know of the cases of diarrhoea and cholera that have struck and are still striking most of the cities and towns in developing countries. Even costs associated with malaria treatment could be reduced if we could reduce breeding places for mosquitoes by keeping our surroundings clean.

Institutional weaknesses, lack of financial resources, improper choice of technology and public apathy (Kaare, 2002; Manga et al., 2008; Oosterveer and Van Vliet, 2010; Okot-Okumu and Nyenje, 2011) are normally cited as major setbacks to proper solid waste management. Rapid urbanisation (Yhdego, 1995; Kyessi and Mwakalinga, 2002, Gawaikar and Deshpande, 2006, Marshal and Farahbakhsh, 2013) is also said to compound the problem. But in the developing world, there is another dimension which is always either overlooked or forgotten – the non-availability of data on the gravity of the problem.

Unlike developed world, the developing world does not, in most cases, have data on how much solid waste is generated or will be generated over a specific period of time. However, in order to marshal the necessary resources (financial, human, equipment and technologies) the authorities need to have a clear picture on the anticipated amount and nature of solid waste - information that is, in most cases, not available. An example is the African Development Bank (2002) report. It can thus be argued that in addition to lack of resources and rapid urbanisation, non-availability of data hampers proper planning. Obviously any proper planning requires the availability of adequate, accurate, reliable and timely data and information. This view is shared by Gawaikar and Deshpande (2006) in their review paper on quantification and characterisation of municipal solid waste in India. Solid waste management systems usually handle large quantities of solid waste and it is therefore important to have detailed information on quantification and characterisation for proper handling of solid waste at different stages of the system. In their paper, Gawaikar and Deshpande (2006) noted that “... there should be reliable information on solid waste quantification and characterisation, which is the prime requirement of the system”. For example, due to non-availability of daily records of quantities and types of solid waste, one cannot assess the total expenditure in disposing the solid waste. So, it is not only the lack of resources that hampers proper solid waste management but also lack of data on the extent of the problem and therefore making it unable to plan properly. “The quantity determines the size and number of functional units and equipment required for managing the waste” (Gawaikar
and quantification is the most important aspect of solid waste management. Furthermore, with data on quantity and type of solid waste, various options can be explored. For example, investment in recycling, reuse, composting, and energy generation could reduce the amount of waste that the authorities have to dispose off and at the same time creating employment. The work of Diaz and Otoma (2013) stress this approach towards minimising solid waste disposed in landfills. Mathematical modelling provides an insight on the extent of the problem by providing data on the quantities generated and their type. Several attempts at modelling solid waste generation have been made. The differences in previous approaches have basically been in the size of the area considered, choice of independent variables, waste streams considered and techniques employed (Beigl et al., 2008). The techniques that have been in use are least square regression (Benitez et al., 2008), linear programming models (Najm et al., 2004) and system dynamics modelling (Dyson and Chang, 2005, Eleyan et al, 2013). The main objective of this present study is to analyse a deterministic compartmental model for the impact of population growth on the solid waste generation and treatment. Stability theory of differential equations is employed to tackle the problem qualitatively. Numerical simulation is performed using Runge-Kutta Fehlberg integration scheme in order to validate the qualitative results. To the best of authors’ knowledge, no such a study which considers this problem has been reported in the literature yet. In the following sections, the model problem is formulated, analysed and solved. Pertinent results are display graphically and discussed qualitatively.

2. Model Development
The model is built on the basic assumption that solid waste generation rate increases with increasing population. The population increases due to birth and migration. It is further assumed that the population is divided into three groups of the young, adults and elderly (A₁, A₂ and A₃ respectively) and the groups generate solid waste at the same rate r. Apart from human activities, solid waste can also be generated from natural causes such trees shedding leaves denoted as K. To keep the environment clean an effort E has to be applied at the rate β. The effort E here means any measures emanating from the government authorities or the community to keep their environment clean. It thus includes efforts geared towards composting, disposal, recycling or reuse. The solid waste diminishes at the rate ρ due to applied effort and decay naturally at the rate δ. Moreover, it is assumed that there is a certain given quantity of solid waste generation (Qₘ) allowed above which the treatment effort E must increase in order to prevent the menace posed by this waste to human health and environment. This implies that dE/dt < 0 whenever Q ≤ Qₘ, while for Q > Qₘ, dE/dt > 0. Thus, by taking p to be the total population increase over a specific period of time and π₁ be the proportion that falls in age group A₁, π₂ the proportion falling in age group A₂ then the fraction that goes to age group A₃ is (1- π₁+ π₂). In this model, the natural death rate is denoted μ for all the three groups. If we take the survival rate of age group A₁ and A₂ to be S₁ and S₂
respectively, then the deterministic compartmental model is as shown in the Figure 1 below:

![Figure 1](image_url)

**Figure 1:** Flow diagram for the compartmental model

Thus, the model is given by the following system of ordinary differential equations:

\[
\frac{dA_1}{dt} = \pi_1 p \tilde{N} - S_1 A_1 - \mu A_1 \\
\frac{dA_2}{dt} = \pi_2 p \tilde{N} + S_1 A_1 - S_2 A_2 - \mu A_2 \\
\frac{dA_3}{dt} = (1 - \pi_1 - \pi_2) P \tilde{N} + S_2 A_2 - \mu A_3 \\
\frac{dQ}{dt} = K + (r \tilde{N} - \delta) Q - \rho E \\
\frac{dE}{dt} = \beta (Q - Q_w) - \gamma E
\]  
(1)-(5)

By adding equations (1)-(3), we obtain

\[
\frac{d\tilde{N}}{dt} = (P - \mu) \tilde{N}
\]  
(6)

where \( \tilde{N} = A_1 + A_2 + A_3 \). Equation (6) models the exponential growth population with \( P \) as the birth / recruitment rate into the population and \( \mu \) as the natural death rate. The intrinsic population growth rate is given by \( P - \mu \). Using equation (6), we normalised equations (1)-(3). Let

\[ a_1 = \frac{A_1}{\tilde{N}}, \quad a_2 = \frac{A_2}{\tilde{N}}, \quad a_3 = \frac{A_3}{\tilde{N}}. \]  
(7)

Substituting equations (6)-(7) into equation (1), i.e.,
\[ \frac{\tilde{N}}{dt} + a_i \frac{d\tilde{N}}{dt} = \pi_i p \tilde{N} - S_i \tilde{N} a_i - \mu \tilde{N} a_i, \]  
\[ \text{(8)} \]

which implies
\[ \frac{\tilde{N}}{dt} + a_i (P - \mu) \tilde{N} = \pi_i p \tilde{N} - S_i \tilde{N} a_i - \mu \tilde{N} a_i. \]
\[ \text{(9)} \]

In a similar manner, equations (6)-(7) are substituted into equations (2)-(3) and the normalised model equations (1)-(3) become
\[ \frac{da_1}{dt} = \pi_1 P - S_1 a_1 - pa_1, \]  
\[ \text{(10)} \]
\[ \frac{da_2}{dt} = \pi_2 P + S_1 a_1 - S_2 a_2 - pa_2, \]  
\[ \text{(11)} \]
\[ \frac{da_3}{dt} = (1 - \pi_1 - \pi_2) P + S_2 a_2 - pa_3. \]  
\[ \text{(12)} \]

Adding equations (10)-(12), we obtain the normalised equation for total population as
\[ \frac{dN}{dt} = P (1 - N), \]  
\[ \text{(14)} \]

and the exact solution is given as
\[ N(t) = 1 - (1 - N_0) e^{-r t}, \]  
\[ \text{(15)} \]

where \( N_0 = a_1(0) + a_2(0) + a_3(0) \) is the total initial population at \( t = 0 \). Substituting equation (15) into equation (4), we obtain the equation for the rate of solid waste accumulation as
\[ \frac{dQ}{dt} = K + r Q [1 - (1 - N_0) e^{-r t}] - \delta Q - \rho E. \]  
\[ \text{(16)} \]

Equation (16) invariably depends on the normalised total population, treatment rate base on applied effort \( E \) and natural decay rate of solid waste. Note that \( r \) is the rate of solid waste generation due to interaction of total population with solid waste materials. Moreover, it is assumed that a certain stipulated quantity of solid waste generation (\( Q_m \)) is allowed by the government authority is a given community above which the waste treatment effort \( E \) must be expedited in order to prevent the menace posed by this waste to human health and environment as depicted in equation (5). This implies that \( dE/dt < 0 \) whenever \( Q \leq Q_m \), while for \( Q > Q_m \) \( dE/dt > 0 \).

**Theorem 1:** If \( a_1(0), a_2(0), a_3(0), Q(0) \) and \( E(0) \), are non-negative, then so are \( a_1(t), a_2(t), a_3(t), Q(t), E(t) \) and \( N(t) = a_1(t) + a_2(t) + a_3(t) \) for all \( t > 0 \). Moreover
\[ \lim_{t \to \infty} N(t) \leq 1, \quad \lim_{t \to \infty} Q(t) \leq Q^*, \quad \lim_{t \to \infty} E(t) \leq E^* \]
where
\[ Q^* = \frac{K}{\gamma (\beta - r) + \rho \beta} Q_m, \quad E^* = \frac{\beta}{\gamma} [Q^* - Q_m] \]
In particular, the region
\[ D = \{(a_1, a_2, a_3, Q, E) \in \mathbb{R}^5_+ : a_1 + a_2 + a_3 \leq 1, Q \leq Q^*, E \leq E^*\} \]
is positively invariant. From this theorem we conclude that it is sufficient to consider the dynamics of equations (10-12, 16, 5) in D. In this region, the model can be considered as being biologically, environmentally and mathematically well-posed.

**Remark:** In order to obtain the solid waste endemic equilibrium \( H_1 = (a_1^*, a_2^*, a_3^*, Q^*, E^*) \), equations (10)-(12), (16) and (5) are equated to zero and solved. The components of \( H_1 \) are given by
\[
\begin{align*}
\hat{a}_1^* &= \frac{\pi_1 P}{S_1 + P}, \\
\hat{a}_2^* &= \frac{\pi_2 P}{S_2 + P} + \frac{S_1 \pi_1 P}{(S_1 + P)(S_2 + P)}, \\
\hat{a}_3^* &= (1 - \pi_1 - \pi_2) + \frac{S_2 \pi_2}{S_2 + P} + \frac{S_1 S_2 \pi_1}{(S_1 + P)(S_2 + P)}, \\
Q^* &= \frac{K \gamma + \rho \beta Q}{\gamma (\delta - r) + \rho \beta}, \\
E^* &= \frac{\beta}{\gamma} [Q^* - Q_0]
\end{align*}
\] (17)

For solid waste free equilibrium \( Q \leq Q_{m0} \) in this case \( H_0 = (a_1^*, a_2^*, a_3^*, 0, 0) \) and virtually all solid wastes automatically decay to zero together with the treatment effort.

The linear stability of \( H_1 \) is investigated by considering the Jacobian matrix of the model system in equations (10)-(12), (16) and (5) given by
\[
J = \frac{\partial (f_1, f_2, f_3, f_4, f_5)}{\partial (a_1, a_2, a_3, Q, E)} =
\begin{bmatrix}
-S_1 - P & 0 & 0 & 0 & 0 \\
S_1 & -S_2 - P & 0 & 0 & 0 \\
0 & S_2 & -P & 0 & 0 \\
0 & 0 & 0 & r - \delta & -\rho \\
0 & 0 & 0 & \beta & -\gamma
\end{bmatrix}
\] (18)

The eigenvalues of the Jacobian matrix in (18) are obtained as
\[
\lambda_1 = -(S_1 + P), \quad \lambda_2 = -(S_2 + P), \quad \lambda_3 = -P,
\] (19)

and
\[
\lambda^2 + \lambda (\gamma - r + \delta) + \beta \rho + \gamma (\delta - r) = 0.
\] (20)

For the local asymptotic stability of \( H_1 \), all the eigenvalues of Jacobian matrix in (18) must have strictly negative real parts. Clearly, the three eigenvalues in equation (19) are strictly negative. However, in order to achieve strictly negative real parts for the remaining two eigenvalues in equation (20), we employed Routh-Hurwitz criteria (Arrowsmith and Place, 1982) which invariably leads to the following additional conditions
\[
(\gamma - r + \delta) > 0, \quad \beta \rho + \gamma (\delta - r) > 0.
\] (21)

From equation (21), we define
\[
R_1 = \frac{\gamma}{r - \delta}, \quad R_2 = \frac{\beta \rho}{\gamma (r - \delta)}.
\] (22)
**Theorem 2:** The solid waste generation endemic equilibrium $H_1$ is locally asymptotically stable whenever $R_1 > 1$ and $R_2 > 1$ and unstable otherwise.

**Remark:** The conditions in theorem 2 validate theorem 1 and the positivity of the solid waste expression $Q^*$ in equation (17). In the absent of solid waste, $H_1$ is unstable and solid waste free equilibrium $H_0$ can be achieved.

3. Results and Discussion

In this section, we study numerically the effects of various parameters embedded in the model system on the solid waste generation and treatment effort in the presence of population growth scenario using Runge-Kutta Fehlberg integration scheme. For numerical simulation, the following parameter values $\gamma = 0.05; \delta = 0.05; \rho = 0.1; K = 0.01; S_1 = 0.1; S_2 = 0.1; \pi_1 = 0.3; \pi_2 = 0.3; P = 0.1, 0.12, 0.15, 0.2; r = 0.1, 0.12, 0.13, 0.15; \beta = 0.1, 0.2, 0.3, 0.4$ are utilised in order to examine the system dynamics. Figures 2-3 depict the population dynamics within the community. The evolution of the three population groups (i.e. young, adults and elderly) are illustrated in figure 2. Each group grow at different rate based on their initial population and the survival probability to the next group with time until they achieve their respective stable population level. In essence, solid waste generation rate for each group will initially increase but eventually be constant as each group comes to a steady state. Figure 3 shows the effects of increasing growth rate on the total population. As expected, the total population increases faster with time due to an increase in the growth rate $(P)$ until the stable population distribution is achieved.

![Figure 2: Variation in population groups with time](image-url)
Figures 4-10 illustrate the effects of various parameters on the solid waste generation and treatment effort needed when the threshold quantity of solid waste generation allowed by the government authority in a given community is $Q_m = 1$. Initially, it is assumed that the quantity of solid waste generated within the community ($Q(0) = 0.5$) is lower than the threshold allowed i.e. ($Q(0) < Q_m$). As the total population increases with time, the solid waste generated also increases gradually above the stipulated threshold as shown in figure 4. Moreover, a further increase in the solid waste generation is observed with increasing population growth rate ($P$). Figure 5 shows the effects of increasing interaction with solid waste material by the community. It is interesting to note that the generation of solid waste by the community increases with increasing rate of access to solid waste materials e.g. plastic bags from supermarkets, electronics, papers, bottles, etc. This may be due to improved economy and standard of living within the community. As $r$ increases, the generated solid waste increases faster above the stipulated threshold. In essence, this calls for an effort to contain the increasing solid waste without which the waste will cause harmful effects to human and the environment. Obviously, solid waste generation decreases as solid waste treatment effort rate increases as shown in Figure 6. This means that efforts geared towards solid waste reuse, recycling, composting and conversion of solid waste to energy or any other means that can reduce waste in the community should be encouraged. Economically, through such efforts raw materials will be recovered, employment opportunities created and most importantly reduce the burden and cost of handling the solid waste. In figure 7, it is observed that effort needed to treat solid waste initially decreases with time since the initial quantity of generated solid waste is lower than the...
threshold quantity. But as the total population continues to increase with increasing growth rate, the solid waste generation continues to increase as well, and grows beyond the threshold quantity. In effect, solid waste treatment effort has to increase in order to reduce the solid waste accumulation. Similarly, figure 8 shows that increasing solid waste generation rate due to access to solid waste materials may initially have no effect on the solid waste treatment effort as some waste may be decaying naturally or the threshold amount has not been reached. But as the solid waste continues to pile up due to increasing access and interaction with solid waste material, the required treatment effort needed to curb the waste generation increases. It is important to note that, in this context, solid waste treatment effort refers to all means, measures or techniques employed to reduce solid waste mount in the community. They may include land-filling, reuse, recycling, composting, incineration and waste to energy conversions or other techniques to that effect. Figure 9 shows an increase in the response to solid waste generation above the stipulated threshold quantity, in time, will lead to a general elevation in the solid waste treatment effort. In effect increasing effort growth rate means reducing the amount of solid waste handled in the community and thus providing healthier environment to live in. From figure 10, it is seen that the treatment effort increases with an increase in the solid waste generation above the threshold quantity, consequently, the quantity of generated solid waste decreases as the effort rate increases.

Figure 4: Effect of increasing population growth rate on solid waste generation
Figure 5: Effect of increasing solid waste generation rate by total population

Figure 6: Effect of increasing treatment effort rate on solid waste generation
**Figure 7:** Effect of increasing population growth rate on treatment effort ($Q_m = 1$)

**Figure 8:** Effect of increasing solid waste generation rate on treatment effort ($Q_m = 1$)
Figure 9: Effect of increasing treatment effort rate when \((Q_m = 1)\)

Figure 10: Effect of increasing treatment effort on solid waste generation.

4. Conclusions
A compartmental model for the solid waste generation and treatment in the presence of population growth, as proposed, tackled qualitatively using stability theory of differential equation and numerically using Runge-Kutta Fehlberg integration scheme. Our results revealed that solid waste generation is enhanced by increasing population growth and access to solid waste materials leading to endemic solid waste equilibrium.
It is also evident that increasing the solid waste treatment results in reduced solid waste quantity. Moreover, below the stipulated threshold quantity of solid waste, the treatment effort diminishes, leading to possibility of solid waste free equilibrium due to natural decay of remnant solid wastes, reuse, recycling, incineration, etc.

REFERENCES